Dielectric Functions and Dispersion Relations of Ultra-Relativistic Plasmas with Collisions

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In the present paper we calculate the dielectric functions of an ultra-relativistic plasma, such as an electron-positron or a quark-gluon plasma. We use classical transport theory and take into account collisions within the relaxation time approximation. From these dielectric functions we derive the dispersion relations of longitudinal and transverse plasma waves.

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I. INTRODUCTION

In this paper, we will study the dielectric functions and dispersion relations for ultra-relativistic plasmas ($m \ll T$). The ultra-relativistic limit is relevant for the study of the quark-gluon plasma [1, 2] or a hot electron-positron plasma, which is present in various astrophysical situations [3] but might also be produced in the laboratory [4]. Dispersion relations are of interest because they describe the propagation of collective plasma waves and provide information about physical quantities like screening lengths, oscillation frequencies, damping and particle production rates, transport coefficients, and the equation of state [5].

Dielectric functions and dispersion relations of ultra-relativistic plasmas have been calculated using transport theory [6] and thermal field theory [7, 8]. The new aspect of the present investigation is that we take into account collisions. For this purpose we start from the Boltzmann equation and use the relaxation time approximation for the collision term together with Maxwell's equations. We follow the non-relativistic derivation presented in Refs.[10, 11]. Note that in a plasma, electromagnetic fields may be produced by external sources. These fields change the distribution and motion of the charged particles in the plasma, creating induced charges and currents, which also produce electromagnetic fields. This system must be analysed self-consistently.

II. MAXWELL'S EQUATIONS

To obtain dispersion relations we start from Maxwell's equations in the plasma ($c = 1$). For this purpose, we introduce the induced carge and current density (ρ, j_i), the complex conductivity σ_{ij}, and the dielectric tensor ϵ_{ij}. Assuming a homogeneous medium these quantities are related to the electric field E_i and electric displacement D_i in momentum space by

$$j_i(\omega,\vec{k}) = \sigma_{ij}(\omega,\vec{k})E_j(\omega,\vec{k}), \qquad (1)$$

$$D_i(\omega,\vec{k}) = \epsilon_{ij}(\omega,\vec{k})E_j(\omega,\vec{k}).$$

In addition we note that the conductivity and the dielectric tensor are related by

$$\epsilon_{ij}(\omega,\vec{k}) = \delta_{ij} + \frac{i}{\omega}\sigma_{ij}(\omega,\vec{k}). \qquad (2)$$

Using these definitions, in the absence of external sources, Maxwell's equations in momentum space become

$$(\vec{k}\times\vec{B})_i = -\omega\,\epsilon_{ij}(\omega,\vec{k})\,E_j, \qquad (3)$$

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$$\vec{k}\cdot\vec{B}=0,$$
$$(\vec{k}\times\vec{E})_i=\omega\,B_i,$$
$$k_i\epsilon_{ij}(\omega,\vec{k})E_j=0.$$

Eliminating $\vec{B}$ we obtain,

$$\left(k^2\delta_{ij}-k_ik_j-\omega^2\epsilon_{ij}\right)E_j=0. \tag{4}$$

We further specialize to an isotropic medium where the dielectric tensor can be decomposed:

$$\epsilon_{ij}(\omega,\vec{k})=P^T_{ij}\epsilon^t(\omega,\vec{k})+P^L_{ij}\epsilon^l(\omega,\vec{k}), \tag{5}$$
$$P^T_{ij}=\delta_{ij}-\frac{k_ik_j}{k^2}\,,\quad P^L_{ij}=\frac{k_ik_j}{k^2},$$
$$\epsilon^t(\omega,\vec{k})=\frac{1}{2}P^T_{ij}\epsilon_{ij}\,,\quad \epsilon^l(\omega,\vec{k})=P^L_{ij}\epsilon_{ij}. \tag{6}$$

Using (5) we have,

$$\left(k^2-\omega^2\epsilon^t\right)\vec{E}_T=0\,,\quad \left(\omega^2\epsilon^l\right)\vec{E}_L=0, \tag{7}$$

where

$$(E_T)_i=P^T_{ij}E_j\,,\quad (E_L)_i=P^L_{ij}E_j. \tag{8}$$

Non-trivial solutions are obtained from

$$\begin{aligned}\epsilon^l(\omega,k)&=0,\\ \omega^2\epsilon^t(\omega,k)-k^2&=0.\end{aligned} \tag{9}$$

III. KINETIC THEORY AND DIELECTRIC FUNCTIONS

Let us consider an ultra-relativistic electron-positron plasma. We use kinetic theory to calculate the quantities ϵ^l and ϵ^t in (9). The essence of kinetic theory is the assumption that the system can be described using a distribution function $f(\vec{p},\vec{r},t)$ which gives the probability density to find a particle of momentum $\vec{p}$ at position $\vec{r}$ at time t. In order to obtain an expression for $f(\vec{p},\vec{r},t)$ we must solve a kinetic equation. For the collisionless plasma it is common to use the Vlasov equation:

$$\frac{\partial f(\vec{p},\vec{r},t)}{\partial t}+\vec{v}\frac{\partial f(\vec{p},\vec{r},t)}{\partial\vec{r}}+e\left[\vec{E}+\frac{1}{c}(\vec{v}\times\vec{B})\right]\frac{\partial f(\vec{p},\vec{r},t)}{\partial\vec{p}}=0. \tag{10}$$

In order to solve this equation we make the approximation that the system is not far from equilibrium and expand around the equilibrium distribution function:

$$f(\vec{p},\vec{r},t)=f_0(p)+\delta f(\vec{p},\vec{r},t)\,,\quad f_0(p)=\frac{4}{e^{p/T}+1}. \tag{11}$$

Note that we have neglected masses in the expression above by taking the ultra-relativistic limit ($m\ll T$). The factor 4 in the equilibrium distribution $f_0(p)$ comes from the 2 spin states and from taking into account electrons and positrons in (10) at the same time, which is possible since the final results depend only on e^2. In addition we define the particle number density

$$N(\vec{r},t)=\int dp f(\vec{p},\vec{r},t)\,,\quad N_0=\int dp f_0(p)\,,\quad dp:=\frac{d^3p}{(2\pi)^3}. \tag{12}$$

In a weakly ionized plasma, we can include the effects of collisions by using a BGK collision term [9, 10, 11], i.e., we use a collision term on the right hand side of (10) of the form

$$-\nu\left[f(\vec{p},\vec{r},t)-N(\vec{r},t)\frac{f_0(p)}{\int dp'\,f_0(p')}\right], \tag{13}$$

where ν is a velocity independent collision frequency. In perturbative QED the collision frequency between electrons, positrons, and photons (elastic $2\to 2$ scattering processes in the Boltzmann collision term) is of order e^4 (see e.g. [12]). Since we are interested in investigating the effect of the collision frequency on the dispersion relations (see below), we will vary the parameter ν from 0 and 1 in units of m_γ, which is the only scale other than T in ultralrelativistic plasmas. (Note that $e\simeq 0.3$ is not far from 1).

The BGK collision term corresponds to an improvement of the relaxation time approximation for the collision term of the Boltzmann equation [18]. It simulates the effect of close binary collisions with substantial momentum transfer [11] and therefore it is particularly useful for describing collisions between ions and neutral gas molecules in a weakly ionized plasma. However, it can also be used as a rough guide for describing collisions between charged particles, such as in a hot electron-positron plasma. Note that the effect of long distance collisions with small momentum transfer can be considered by using a mean-field electric field $\vec{E}$ in (10) [13].

We substitute (11) into (10) and (13) and linearize in the deviation from equilibrium δf to obtain,

$$\delta f(\vec{p},\vec{r},t)=\left[-ie\vec{E}\cdot\frac{\partial f_0(p)}{\partial\vec{p}}+i\nu\eta f_0(p)\right]D^{-1}(p), \tag{14}$$

where

$$\eta:=\frac{1}{N_0}\int dp\,\delta f(\vec{p},\vec{r},t)\,,\quad D(p):=\omega+i\nu-\vec{k}\cdot\vec{v}(p)\,,\quad \vec{v}(p)=\frac{\vec{p}}{p}. \tag{15}$$

Solving (14) and (15) for $\delta f(\vec{p},\vec{r},t)$ we obtain an integral expression for the induced current:

$$\begin{aligned}
j_i &= e\int dp\,v_i(p)\,\delta f(\vec{p},\vec{r},t)\\
&= -ie^2\int dp\,v_i(p)\,\frac{(\vec{E}\cdot\vec{p})}{p}\frac{\partial f_0}{\partial p}D^{-1}(p)\\
&+\frac{e^2\nu}{N_0}\int dp\,v_i(p)f_0(p)D^{-1}(p)\int dp'\frac{(\vec{E}\cdot\vec{p}')}{p'}\frac{\partial f_0(p')}{\partial p'}D^{-1}(p')\left[1-\frac{i\nu}{N_0}\int dp''f_0(p'')D^{-1}(p'')\right]^{-1}.
\end{aligned} \tag{16}$$

Using (1) we extract an expression for $\sigma_{ij}(\omega,\vec{k})$:

$$\begin{aligned}
\sigma_{ij} &= -ie^2\int dp\;v_i(p)v_j(p)\frac{\partial f_0(p)}{\partial p}D^{-1}(p)\\
&+\frac{e^2\nu}{N_0}\int dp\,v_i(p)f_0(p)D^{-1}(p)\int dp'v_j(p')\frac{\partial f_0(p')}{\partial p'}D^{-1}(p')\left[1-\frac{i\nu}{N_0}\int dp''f_0(p'')D^{-1}(p'')\right]^{-1}.
\end{aligned} \tag{17}$$

Using (2) and (5) we obtain:

$$\begin{aligned}
\epsilon^l &= 1+\frac{i}{\omega k^2}(k_ik_j\sigma_{ij}),\\
\epsilon^t &= \frac{3}{2}+\frac{i}{2\omega}(\sigma_{ii})-\frac{\epsilon^l}{2}.
\end{aligned} \tag{18}$$

In the ultra-relativistic case we take $v=1$ which means that in (17) the radial integral over p and the angular integrals decouple. Thus, in contrast to the non-relativistic case [10], we are able to perform an analytic calculation of the dielectric functions. We define the integrals:

$$\begin{aligned}
I_1 &= \int_0^\infty dy\,\frac{y}{e^y+1}=\frac{\pi^2}{12},\\
J_1 &= \int_{-1}^1 dx\,\frac{x^2}{z-x}=-z\,X(z),
\end{aligned}$$

$$J_2 = \int_{-1}^{1} dx \, \frac{x}{z-x} = -X(z),$$
$$J_3 = \int_{-1}^{1} dx \, \frac{1}{z-x} = \frac{2-X(z)}{z}, \tag{19}$$

where $z = (\omega + i\nu)/k$ and $X(z) = 2 - z \ln\left(\frac{z+1}{z-1}\right)$. Using the definitions of $f_0(p)$ and $D(P)$ given in (11) and (15) we obtain:

$$\epsilon^l = 1 - \frac{2e^2T^2}{\pi^2 \omega k} I_1 \left(J_1 + \frac{i\nu}{2k} \frac{J_2^2}{(1 - \frac{i\nu}{2k} J_3)} \right),$$
$$\epsilon^t = 1 - \frac{e^2T^2}{\pi^2 \omega k} I_1 \left(J_3 - J_1 \right). \tag{20}$$

Substituting in from (19) and defining the effective photon mass $m_\gamma = eT/3$ we obtain:

$$\epsilon^l(\omega,k) \;=\; 1 + \frac{3m_\gamma^2}{k^2} \left(1 - \frac{\omega + i\nu}{2k} \ln \frac{\omega + i\nu + k}{\omega + i\nu - k} \right) \left(1 - \frac{i\nu}{2k} \ln \frac{\omega + i\nu + k}{\omega + i\nu - k} \right)^{-1}, \tag{21}$$
$$\epsilon^t(\omega,k) \;=\; 1 - \frac{3m_\gamma^2}{2\omega(\omega + i\nu)} \left\{ 1 + \left[\frac{(\omega+i\nu)^2}{k^2} - 1 \right] \left(1 - \frac{\omega + i\nu}{2k} \ln \frac{\omega + i\nu + k}{\omega + i\nu - k} \right) \right\}.$$

Let us discuss the analytic structure of these dielectric functions by comparing them with the collisionless results.

(1). In the collisionless case, $\nu = 0$, (21) reduces to the expressions found by using the Vlasov equation [6], or the high-temperature limit of thermal QED [7, 8]. The correction to the vacuum value $\epsilon^{l,t} = 1$ is proportional to the square of the effective photon mass, i.e., of order e^2T^2.

(2). Note that the longitudinal dielectric function does not follow from the collisionless case by simply replacing ω by $\omega + i\nu$. As a consequence, the longitudinal part of the polarization tensor, $\Pi^l = k^2(1 - \epsilon^l)$ does not depend only on $(\omega + i\nu)/k$, but also on $i\nu/k$ separately. The transverse part of the polarization tensor, on the other hand, $\Pi^t = \omega^2(1 - \epsilon^t)$, depends only on $(\omega + i\nu)/k$.

(3). Due to collisions, the dielectric functions (21) have explicit imaginary parts from terms containing $i\nu$. In addition, even for $\nu = 0$, there are imaginary parts coming from the logarithms, which are related to Landau damping.

Apart from the constant factor m_γ, (21) agrees with the results for the dielectric functions of a non-relativistic degenerate plasma as given in [10]. This agreement occurs because the radial and angular integrals decouple for both the relativistic and non-relativistic degenerate plasmas, and the angular integrals, giving the functional dependence on ω and k, are identical in both cases. Note that for $\omega = 0$, the longitudinal dielectric function becomes independent of the collision rate ν. Therefore, the (square of the) Debye screening mass $m_D^2 = k^2[\epsilon^l(\omega = 0) - 1] = 3m_\gamma^2$ is also independent of ν.

IV. DISPERSION RELATIONS

The equations (9) and (21) must be solved numerically. We scale all dimensionful variables by m_γ and produce numerical results for ω/m_γ versus k/m_γ for various values of ν/m_γ. Figs. (1-4) show the real and imaginary parts of the dispersion relations for transverse and longitudinal modes. We correctly reproduce the known results from the Vlasov equation (10) in the limit $\nu = 0$ [6]. These results can also be derived from quantum field theory, using lowest order perturbation theory in the high-temperature or hard thermal loop approximation [7, 8, 14]. The real part of the dispersion relation is remarkably insensitive to ν. The imaginary part, which is identically zero when $\nu = 0$, acquires a non-zero contribution for $\nu \neq 0$.

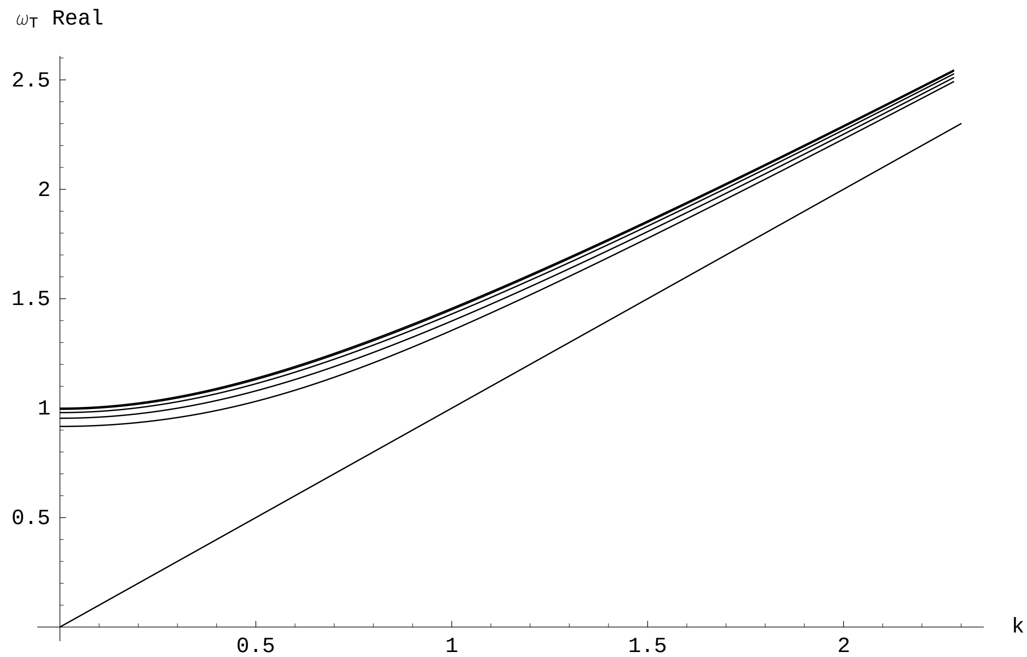

FIG. 1: Real part of the transverse dispersion relation (the parameter ν runs from 0 to 0.8 in steps of 0.2 with the smallest value being at the top of the graph)
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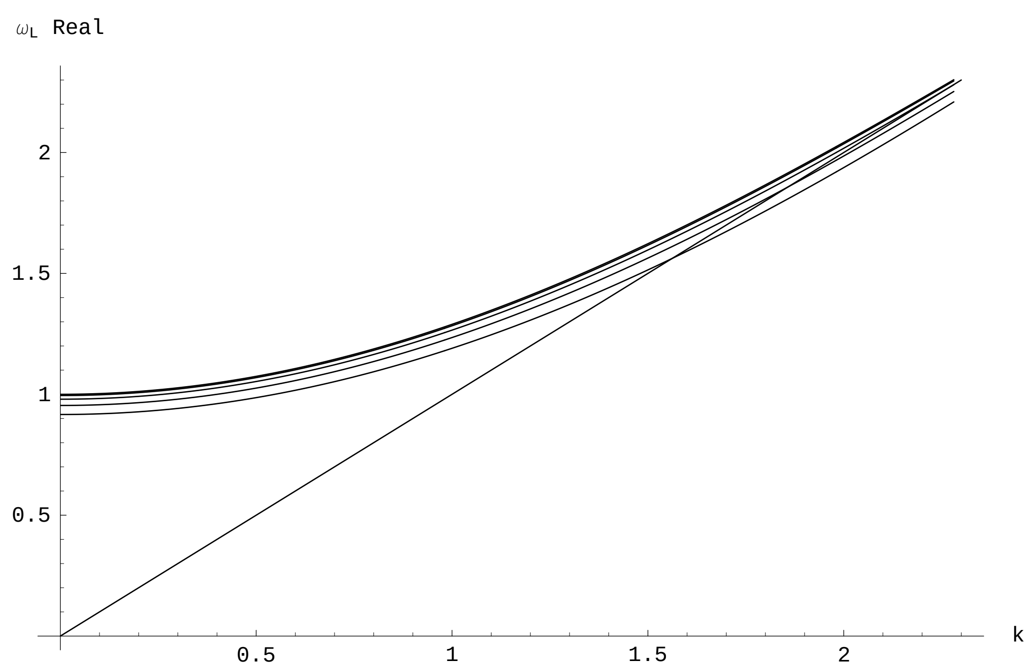

FIG. 2: Real part of the longitudinal dispersion relation (the parameter ν runs from 0 to 0.8 in steps of 0.2 with the smallest value being at the top of the graph)
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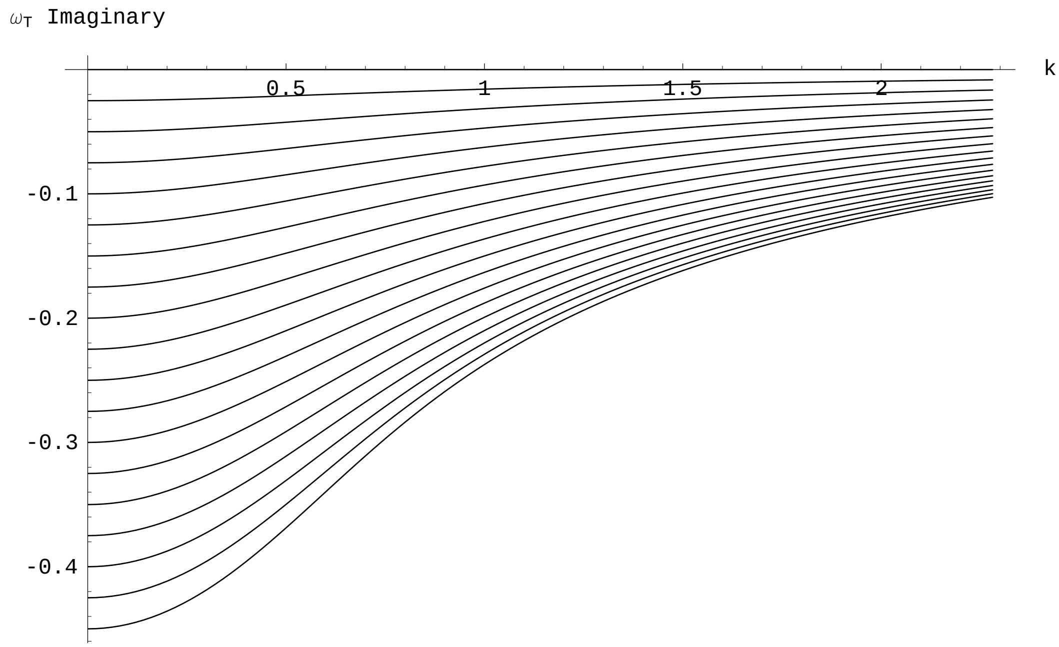

FIG. 3: Imaginary part of the transverse dispersion relation (the parameter ν runs from 0 to 0.9 in steps of 0.05 with the smallest value being at the top of the graph)
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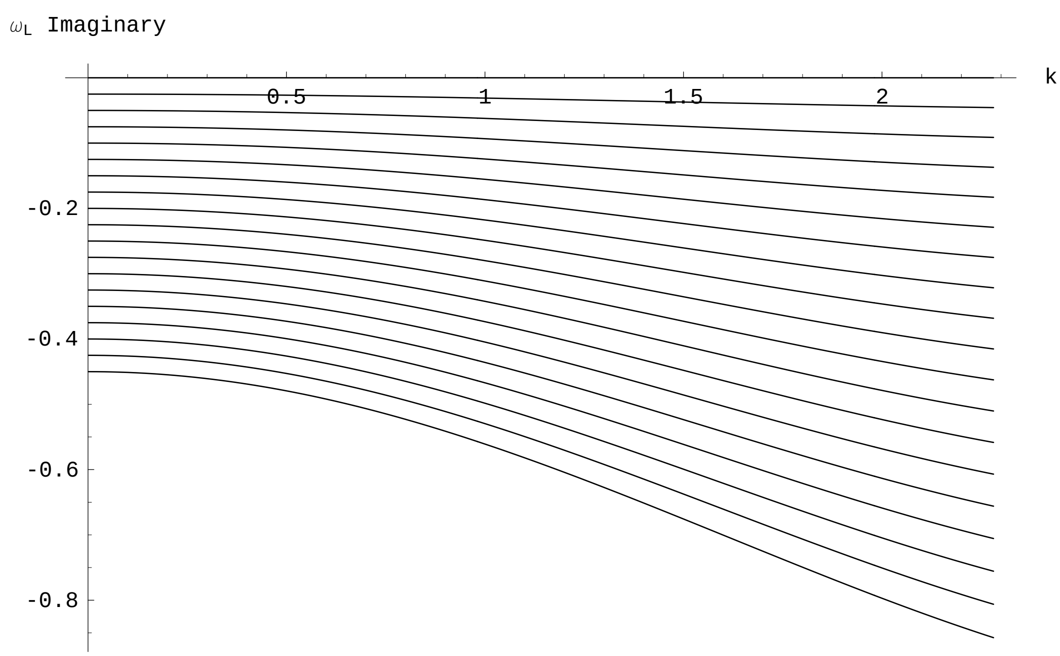

FIG. 4: Imaginary part of the longitudinal dispersion relation (the parameter ν runs from 0 to 0.9 in steps of 0.05 with the smallest value being at the top of the graph)
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V. DISCUSSIONS AND CONCLUSIONS

In the present investigation we have calculated the dielectric functions and the dispersion relations of electromagnetic plasma waves in an ultra-relativistic electron-positron plasma, taking collisions into account. Adding the BGK collision term to the Vlasov equation, we have derived analytic expressions for the longitudinal and transverse dielectric functions in the case of a small deviation from equilibrium. Using Maxwell's equations, the dispersion relations of longitudinal and transverse plasma waves follow numerically from the dielectric functions. Surprisingly, the deviations from the collisionless case are rather small, resulting only in a small shift of the dispersion relations towards lower energies. For example, the plasma frequency, given by $\omega_L(k=0)=\omega_T(k=0)$, is reduced from m_γ at $\nu=0$ to only about $0.9\,m_\gamma$ at $\nu=m_\gamma$ [15]. However, when collisions are included, the longitudinal dispersion at $\nu\neq 0$ intersects the light cone $\omega=k$, in contrast to the case of collisionless dispersion, where $\omega_{L,T}(k)>k$ for all k. In the collisionless case the only damping mechanism is Landau damping, which occurs for $\omega^2<k^2$, therefore plasma waves are undamped for $\nu=0$. For $\nu\neq 0$ the plasma waves are damped by the collisions, which produces damping rates $\gamma_{L,T}(k)=-\mathrm{Im}\,\omega_{L,T}(k)$. The longitudinal wave is also Landau damped for momenta that satisfy $\omega_L(k)<k$.

It should be noted that we have introduced the collision rate ν phenomenologically as a constant free parameter. In hydrodynamical models one normally assumes that local equilibrium is achieved after some momentum independent relaxation time [16], which can be identified with the inverse of the collision rate. In principle, the collision rate could be determined perturbatively leading to a momentum dependent rate proportional to e^4 [12]. However, it would be much more difficult to work with a momentum dependent collision rate.

The effect of collisions in a quark-gluon plasma, e.g., on the damping of plasma oscillations and energetic quarks and gluons, has been discussed extensively in the literature using perturbative QCD as well as transport theory (see e.g. [12, 17]). Our results for the dielectric functions as well as the dispersion relations are valid for chromo-electromagnetic plasma waves in a quark-gluon plasma, if we replace the effective photon mass m_γ by an effective gluon mass $m_g=gT\sqrt{(1+n_f/6)/3}$, where n_f is the number of quark flavors with masses $m\ll T$ [5]. However, this result is not fully consistent: neglecting the non-abelian terms of order g in the Yang-Mills equations contradicts the asumption that we can use a phenomenological collision term to model all collision processes, including higher order processes in g. At best, our phenomenological approach produces a rough estimate of the effect of collisions on plasma waves in a quark-gluon plasma.

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